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Letters and comments

Comment on 'An important question about rock climbing' by G Reali and L Stefanini 1996 *Eur. J. Phys.* 17 348–52

Abstract. The original article discussed the heat generated in a friction device as a consequence of arresting a falling climber. Here we point out that, on the short timescale of such an event, heat generation would occur almost entirely at the interface between the rope surface and the metal friction device, leading to unacceptably high temperatures and possible rope melting.

Résumé. Le premier article discutait le phénomène de chaleur générait par moyen de friction suite à l'arrêt de la chute d'un ascensionniste. Ici, nous mettons en exergue le fait que étant donné la courte durée d'un tel événement, la génération de chaleur interviendrait presque complétement à l'interface entre la surface de la corde et l'appareil de friction métallique, conduisant à des températures inacceptablement hautes et à la possible fusion de la corde.

This paper will certainly be of interest to all rock climbers, whether they be experienced physicists or not. It is concerned with the need to limit the maximum force experienced by a falling climber when being brought to rest by the rope. The conclusion is that in order to limit the deceleration forces, a short length of rope should be allowed to slide rapidly through the friction device, thereby dissipating some of the kinetic energy of the fall as heat.

The authors calculate that in order to limit the deceleration of the fall to 5 g, the energy dissipated thermally in the typical fall being discussed would be around 19 kJ. The purpose of this note is to point out that 19 kJ being dissipated in the metal of the friction device would raise its temperature by some 200°C (for a 100 gm aluminium device). This would not be a cause for concern, were it not for another factor. Since this heat would be generated at the interface between the rope and the metal in a time order of 0.1 s (the typical timescale in figures 1–3 of their article), the finite thermal diffusivity of the metal would limit the temperature rise to the small thickness of metal immediately adjacent to the rope, thereby producing a much higher temperature *and inevitably causing the nylon rope to melt*. Considerations based on thermal diffusivity also mean that this problem would not be overcome by using a greater mass of metal in the friction device (though an increase in the contact area between rope and metal would help matters). The suggested procedure could therefore lead to a dangerous and potentially fatal situation.

In this context, one of us recalls the advice given during a rock climbing training course some years ago: When using a *Sticht plate* (the friction device in vogue then) the maximum descent speed on a free abseil was never to be more than $1\,\mathrm{m\,s^{-1}}$. A story was told of a climber who descended rapidly on a free abseil, but then stopped half way down to look at the view; inevitably, the heat he had generated in his Sticht plate was sufficient to melt through his rope at that point, and he fell to his death.

We appreciate the author's wish to limit the damage caused by a fall in rock climbing, but since one of us has survived a 30 m fall at a factor of 1.5 or thereabouts, the effects of sudden deceleration are perhaps less serious than the authors believe.

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Analysis of the spreading Gaussian wavepacket

Abstract. The $\psi(x,t)$ wavefunction of a Gaussian wavepacket spreading in free space ($V(x)\equiv 0$) is expressed in a didactic form. The expression found is a product of pure real factors and pure phase factors. This makes it very easy to derive the expression for the probability density from the wavefunction. The physical meaning of each of the factors is analysed.

Zusammenfassung.
Analyse der Verbreiterung von Gaußschen Wellenpaketen. Die Wellenfunktion $\psi(x,t)$ eines Gaußschen Wellenpaketes, welches sich bei der Ausbreitung im freien Raum ($V(x)\equiv 0$) verbreitert, wird in einer didaktischen Form ausgedrückt. Der gefundene Ausdruck ist ein Produkt von reellen Faktoren und reinen Phasen-Faktoren. Dies vereinfacht die Herleitung des Ausdruckes für die Wahrscheinlichkeitsdichte aus der Wellenfunktion. Die physikalische Bedeutung von jedem Faktor wird analysiert.

Introduction

While writing a paper [1] about the time evolution of different wavepackets I wanted to find a didactic expression for the $\psi(x,t)$ wavefunction of a *Gaussian* initial state. The particular expression that was finally constructed is different from those found in quantum mechanics texts [2, 3].

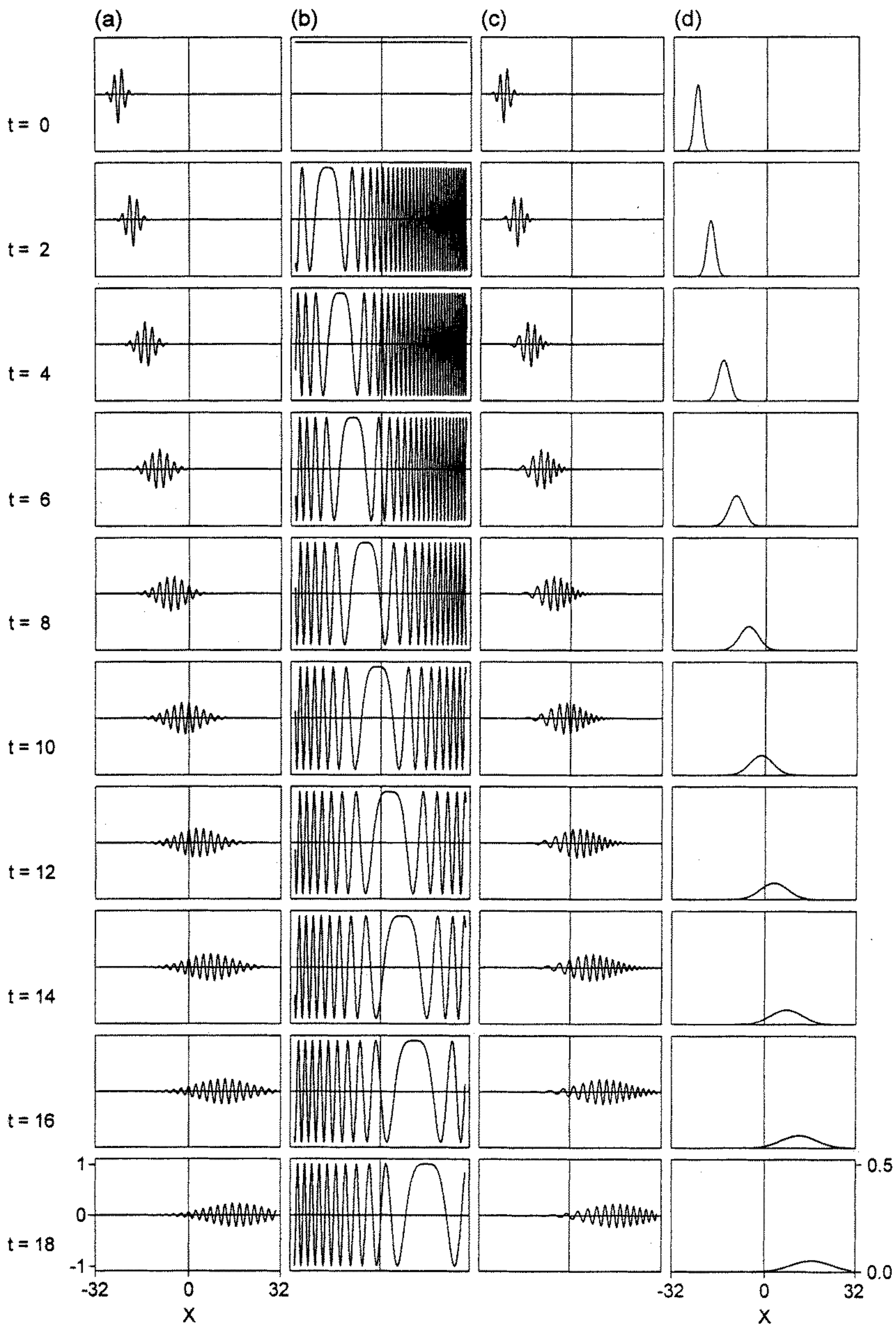

Figure 1. x- and t-dependent parts of the wavefunction. (a), (b) and (c) show the time development of the real part of *factor 1*, *factor 2* and of the full $\psi(x,t)$. The x, y scale is the same for all (a), (b) and (c). (d) shows the time development of the probability density $\rho(x,t)$. The x, y scale is the same for all time instants. *Atomic units* ($\hbar = m_e = 1$) are used. $a = 2.5$ Bohr = 0.13 nm, $\lambda = h/p_0 = 8/3$ Bohr = 0.14 nm. The atomic time unit is 2.41×10^{-17} s. See the text for details.

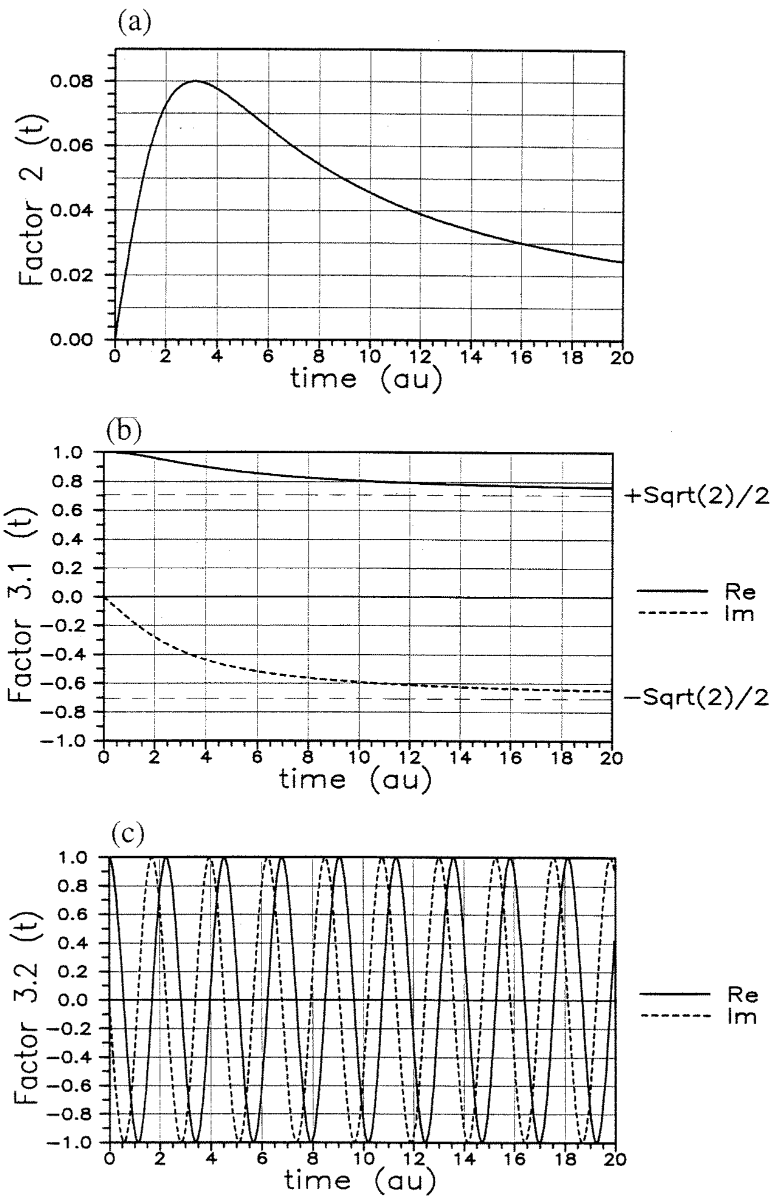

Figure 2. *t*-dependent parts of the wavefunction. (a) shows the time dependent prefactor of *factor 2* as function of time. (b) and (c) show the time dependence of the terms of *factor 3*. Their real (full curve) and imaginary (broken curves) parts are plotted against time. The thin dashed horizontal lines in (b) show the asymptotes for $t = \infty$. *Atomic units* ($\hbar = m_e = 1$) are used. See the text for details.

Initial state

Our initial state is a simple *Gaussian* wavepacket of the form

$$\psi_0(x; a, x_0, p_0) = \sqrt[4]{\frac{2}{\pi a^2}} \exp\left(\mathrm{i}\frac{p_0}{\hbar}x\right) \times \exp\left(-\frac{(x - x_0)^2}{a^2}\right). \qquad (1)$$

This wavepacket is a product of three factors:

A normalization factor that makes the norm $\int_{-\infty}^{+\infty} |\psi(x)|^2 \,\mathrm{d}x$ of the wavefunction unity.

A plane wave factor that accounts for the non-zero momentum p_0 of the wavepacket.

A bell-shaped localizing function with half width at half maximum $\sqrt{\ln 2}a$.

Time evolution

The time development of the initial $\psi_0(x)$ state is given by [2]:

$$\psi(x,t) = \mathcal{F}^{-1}\left[\mathrm{g}(k)\exp\left(-\mathrm{i}\frac{k^2}{2}t\right)\right](x) \tag{2}$$

$$\mathrm{g}(k) = \mathcal{F}\left[\psi_0(x)\right](k). \tag{3}$$

This *Fourier integral* can be calculated easily with Gaussian integrals and leads to a wavefunction like

$$\begin{aligned}\psi(x,t) = &\sqrt[4]{\frac{2}{\pi a^2(1+2\mathrm{i}(\hbar/m)t/a^2)^2}}\\ &\times\exp\{[-(x-x_0)^2 - 2(p_0/m)x_0t + \mathrm{i}a^2(p_0/\hbar)x\\ &-\mathrm{i}a^2(p_0^2/(2m)/\hbar)t][a^2(1+2\mathrm{i}(\hbar/m)t/a^2)]^{-1}\}.\end{aligned} \tag{4}$$

Transformation of $\psi(x,t)$ into didactic form

Now we want to transform this into something more informative. First note that the centre of the wavepacket is moving with the *group velocity* $u_\mathrm{g} = p_0/m$. Hence it is worth writing $x_0 + (p_0/m)t$ instead of x_0 into the first term of the numerator in the exponential. Working this out gives the following result

$$\begin{aligned}\psi(x,t) = &\sqrt[4]{\frac{2}{\pi a^2(1+2\mathrm{i}(\hbar/m)t/a^2)^2}}\\ &\times\exp\left(-\frac{[x-(x_0+(p_0/m)t)]^2}{a^2(1+2\mathrm{i}(\hbar/m)t/a^2)}\right)\\ &\times\exp\left(\mathrm{i}\frac{p_0}{\hbar}x\right)\exp\left(-\mathrm{i}\frac{p_0^2}{2m}\frac{1}{\hbar}t\right).\end{aligned} \tag{5}$$

It is getting clearer already! Now let us get rid of the complex denominators!

$$\frac{1}{1+2\mathrm{i}(\hbar/m)t/a^2} = \frac{1-2\mathrm{i}(\hbar/m)t/a^2}{1+4(\hbar^2/m^2)t^2/a^4}. \tag{6}$$

Utilizing this we get finally

$$\psi(x,t) = \sqrt[4]{\frac{2}{\pi|a(t)|^2}}\exp\left(-\frac{[x-(x_0+(p_0/m)t)]^2}{|a(t)|^2}\right)$$

$$\times\exp\left(\mathrm{i}\frac{p_0}{\hbar}x\right) \tag{7}$$

$$\times\exp\left(2\mathrm{i}\frac{\hbar}{m}\frac{t}{a^2}\frac{[x-(x_0+(p_0/m)t)]^2}{|a(t)|^2}\right) \tag{8}$$

$$\times\exp\left(-\frac{\mathrm{i}}{2}\arg a(t)\right)\exp\left(-\mathrm{i}\frac{p_0^2}{2m}\frac{1}{\hbar}t\right) \tag{9}$$

$$a(t) = a + 2\mathrm{i}\frac{\hbar}{m}\frac{t}{a} \tag{10}$$

where $\arg z$ is the phase of the complex number z, i.e. $\arg R\,\mathrm{e}^{\mathrm{i}\varphi} = \varphi$. Our $\psi(x,t)$ has three main factors ((7), (8) and (9)). The first factor (7) is a product of two *pure real* coefficients and a plane wave. This plane wave part of *factor 1* and the entire second (8) and third (9) factors are pure *phase factors*, i.e. their magnitude is one. Hence it is very easy to calculate the probability density $\rho(x,t) = |\psi(x,t)|^2$; one has only to calculate the square of the pure real coefficients of *factor 1* which gives:

$$\rho(x,t) = \sqrt{\frac{2}{\pi|a(t)|^2}}\exp\left(-2\frac{[x-(x_0+(p_0/m)t)]^2}{|a(t)|^2}\right). \tag{11}$$

The three terms of $\psi(x,t)$ are as follows.

Factor 1. (Cf (7)) A Gaussian of the form (1). This is an expression having the same form as $\psi_0(x)$ but the centre of gravity of the Gaussian is moving with speed $u_\mathrm{g} = p_0/m$ and its width is increased to $|a(t)| = \sqrt{a^2 + 4(\hbar^2/m^2)t^2/a^2}$. The maximum value of the Gaussian is decreasing as its width increases making the area under $\rho(x,t)$ (total probability) constant (one). The time evolution of *factor 1* is shown in figure 1(a).

Factor 2. (Cf (8)) An x- and t-dependent phase factor that is quadratic in x. One can see from figure 1(b) that this factor oscillates faster for larger $|x|$ values. This accounts for the fact that the higher momentum components of the initial Gaussian $\psi_0(x)$ move with higher velocities. The function which describes the time-dependent prefactor of the phase is $2t/(a|a(t)|)^2$. This function (cf figure 2(a)) is not monotonic in time. Its value is zero for $t = 0$ and $t = \infty$ and has a maximum at $t = (m/\hbar)a^2/2$.

Factor 3. (Cf (9)) An x-independent (but still t-dependent) phase factor. This phase factor is a product of two terms. The first term is a monotonic function of time while the second one is oscillating. The phase of the first term is zero for $t = 0$ ($a(t)$ is pure real) and $-\pi/4$ for $t = \infty$ ($a(t)$ is pure imaginary). The second term is $\exp(-\mathrm{i}\omega_0 t)$ where $\omega_0 = E/\hbar = p_0^2/(2m)/\hbar$ and it accounts for the time development of the plane wave component $\exp((\mathrm{i}p_0/\hbar)x)$ in *factor 1*. These two phase factors are plotted in figures 2(b) and 2(c) against time.

Acknowledgment

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